J Glob Optim (2008) 40:739–749
DOI 10.1007/s10898-006-9112-2

ORIGINAL PAPER

On system of generalized vector variational inequalities

S. H. Hou · H. Yu · G. Y. Chen

Received: 17 October 2006 / Accepted: 25 October 2006 / Published online: 6 December 2006

Abstract In this paper, we introduce a new system of generalized vector variational inequalities with variable preference. This extends the model of system of generalized variational inequalities due to Pang and Konnov independently as well as system of vector equilibrium problems due to Ansari, Schaible and Yao. We establish existence of solutions to the new system under weaker conditions that include a new partial diagonally convexity and a weaker notion than continuity. As applications, we derive existence results for both systems of vector variational-like inequalities and vector optimization problems with variable preference.

Keywords Generalized vector variational inequalities · Partial diagonally quasi-convexity · Variational-like inequalities · Variable preference

1 Introduction

In [16] Pang introduced a system of (scalar) variational inequalities (specifically, an asymmetric variational inequality problems over product sets) which could uniformly model several equilibrium problems such as traffic equilibrium problems, spatial equilibrium problems, Nash equilibrium problems and equilibrium programming problems, etc. Pang [16] and subsequently Cohen and Chaplais [9] studied the computation of this model, while existence results were established by Bianchi [6] as well as

S. H. Hou (✉)
Department of Applied Mathematics, The Hong Kong Polytechnic University
Hong Kong, China
e-mail: mahoush@polyu.edu.hk

H. Yu
College of Economics and Business Administration, Chongqing University, Chongqing 400044, China

G. Y. Chen
Institute of Systems Science, Chinese Academy of Science, Beijing, China

Ansari and Yao [1]. Further developments on this problem are discussed, for example, in Konnov [13], where system of generalized variational inequalities was tackled in order to solve vector optimization problem, and in Ansari and Yao [3] where existence result for system of generalized inequalities was established. Moreover, Ansari et al. [4] studied the system of vector equilibrium problems that includes, as special case, the system of vector variational inequalities that is the vector generalization of the system of scalar variational inequalities.

In this paper, we introduce systems of weak and strong vector variational inequalities with variable preference. These not only include all the above-mentioned system models as special cases, but also could be considered as a systematic generalization of existing models for vector variational inequalities with variable preference (see [5,10] and references therein) and vector equilibrium problem with variable preference (see [7,8,11,14,15,17,18] and references therein).

Throughout this paper, we denote by co$A$, int A the convex hull and the interior of a set A in a topological vector space, respectively.

Let I be a given index set. For each $i \in I$, let X_i and Y_i be nonempty subsets of two locally convex Hausdorff topological spaces, and E_i be a Hausdorff topological vector space. We also write

$$X = \prod_{i \in I} X_i, \qquad Y = \prod_{i \in I} Y_i \quad \text{and} \quad E = \prod_{i \in I} E_i.$$

An element of the set $X^i := \prod_{j \in I \setminus \{i\}} X_j$ will be denoted by x^i. Thus, we can write $x = (x^i, x_i) \in X^i \times X_i = X$ for all $x \in X$. Let $\{f_i\}_{i \in I}$ be a family of trifunctions defined on $X \times X_i \times Y_i$ with values in E_i. Let $C_i: X_i \to 2^{E_i}$ and $T_i: X \to 2^{Y_i}$ be set-valued maps.

We consider the following two system models:

(1) The system of weak generalized vector variational inequality problem, in short SWGVVI, is to find $\bar{x} \in X$ and $\bar{y}_i \in T_i(\bar{x}_i)$ for each $i \in I$, such that

$$f_i(\bar{x}, z_i, \bar{y}_i) \notin -\text{int}\, C_i(\bar{x}_i) \quad \forall z_i \in X_i.$$

(2) The system of strong generalized vector variational inequality problem, in short SSGVVI, is to find $\bar{x} \in X$ and $\bar{y}_i \in T_i(\bar{x}_i)$ for each $i \in I$, such that

$$f_i(\bar{x}, z_i, \bar{y}_i) \in C_i(\bar{x}_i), \quad \forall z_i \in X_i.$$

The rest of the paper is organized as follows. In Sect. 2, we give a short account of set-valued maps and their properties. In particular, we introduce the new concept of partial diagonally quasi-convexity together with its relationship to other convexity properties. In Sect. 3, the existence results of SWGVVI and SSGVVI are established. Finally, we conclude with some applications in Sect. 4.

2 Preliminaries

Definition 2.1 A set-valued map $F: X \to 2^Y$ is said to have open lower sections if the set $F^{-1}(y) := \{x \in X: y \in F(x)\}$ is open in X for every $y \in Y$.

Lemma 2.1 (Ansari [2]) *Let X be a topological space and Y be a convex subset of a topological vector space. Let $G: X \to 2^Y$ be a set-valued map with open lower sections.*

Then the set-valued map $M: X \to 2^Y$, defined by $M(x) := \mathrm{co}(G(x))$ for all $x \in X$, has open lower sections.

Lemma 2.2 (Klein and Thompson [12], Theorem 8.1.3, p. 97) *Let X be a compact Hausdorff space and let $f: X \to 2^Y$ be a set-valued map from X to a topological vector space Y with convex values. Suppose further that f has open lower sections. Then f has a continuous selection.*

Remark 2.1 The proof of Lemma 2.2 given in [12] can be modified to hold even for X paracompact.

We mention also here the well known Kakutani-Fan's fixed point theorem.

Lemma 2.3 *Let X be a nonempty compact convex subset of a locally convex Hausdorff topological vector space E. Suppose that $H: X \to 2^X$ is a upper semi-continuous set-valued map with nonempty closed convex values. Then H has a fixed point in X.*

We now introduce the following new notion of partial quasi-convexity property which plays a crucial role in our analysis and discussion below.

Definition 2.2 Let $C_i: X_i \to 2^{E_i}$ and $g: X \times X_i \to E_i$ be given. $g(x, z_i)$ is said to be weak (resp. strong) partial diagonally quasi-convex in z_i with respect to C_i if for any $\Lambda = \{x_{i_1}, x_{i_2}, \ldots, x_{i_n}\}$ in X_i and for any $x = (x^i, x_i) \in X$ with $x_i \in \mathrm{co}\Lambda$, there exists a $j \in \{1, 2, \ldots, n\}$ such that

$$g(x, x_{i_j}) \notin -\mathrm{int}\, C_i(x_i), \qquad (\text{resp. } g(x, x_{i_j}) \in C_i(x_i)).$$

Remark 2.2 When $I = \{1\}$, $X_1 = X$, $C_1(x) = \mathbb{R}^+$ for each $x \in X$, and $g: X \times X \to \mathbb{R}$ is a single-valued function, then the above two kinds of partial diagonally quasi-convexity all reduce to 0-diagonally quasi-convexity for scalar functions in Zhou and Chen [19].

Remark 2.3 Strong partial diagonally quasi-convexity with respect to C_i implies weak partial diagonally quasi-convexity with respect to C_i if for all $x_i \in X_i$, $C_i(x_i)$ is a pointed cone with $\mathrm{int}\, C(x_i) \neq \emptyset$.

3 Existence results

We are now ready to prove the following existence results of solutions for SWGVVI.

Theorem 3.1 *For each $i \in I$, let X_i and Y_i be nonempty compact convex metrizable subsets of two locally convex Hausdorff topological vector spaces, and E_i be a Hausdorff topological vector space. Let $f_i: X \times X_i \times Y_i \to E_i$ be a trifunction, $C_i: X_i \to 2^{E_i}$ be a set-valued map with $\mathrm{int}\, C_i(x_i) \neq \emptyset$ for all $x_i \in X_i$, and $T_i: X \to 2^{Y_i}$ be a upper semi-continuous map with nonempty closed convex values.*

Assume that the following conditions are satisfied.

(1) *For each $i \in I$ and for each $y_i \in Y_i$, the function $f_i(x, z_i, y_i)$ is weak partial diagonally quasi-convex in z_i with respect to C_i.*

(2) *For each $i \in I$ and $z_i \in X_i$, the set*

$$\{(x, y_i) \in X \times Y_i : f_i(x, z_i, y_i) \in -\mathrm{int}\, C_i(x_i)\}$$

is open.

Then, there exist $\bar{x} \in X$ *and* $\bar{y}_i \in T_i(\bar{x})$ *for each* $i \in I$*, such that*

$$f_i(\bar{x}, z_i, \bar{y}_i) \notin -\mathrm{int}\ C_i(\bar{x}_i), \quad \forall z_i \in X_i.$$

Proof Define, for each $i \in I$, a set valued map $P_i: X \times Y_i \to 2^{X_i}$ by

$$P_i(x, y_i) := \{z_i \in X_i: f_i(x, z_i, y_i) \in -\mathrm{int}\ C_i(x_i)\}, \quad \forall (x, y_i) \in X \times Y_i.$$

By condition (2), P_i has open lower sections and so does the set-valued map $\varphi_i: X \times Y_i \to 2^{X_i}$ with

$$\varphi_i(x, y_i) := \mathrm{co}P_i(x, y_i), \quad \forall (x, y_i) \in X \times Y_i.$$

Let

$$W_i := \{(x, y_i) \in X \times Y_i: \varphi_i(x, y_i) \neq \emptyset\}.$$

The fact that φ_i has open lower sections implies W_i is open. Moreover, since X and Y_i are metrizable spaces, W_i is paracompact. The restriction of φ_i to W_i has nonempty convex values and open lower sections. It follows from Lemma 2.2 and Remark 2.1 that there exists a continuous selection $s_i: W_i \to X_i$ such that $s_i(x, y_i) \in \varphi_i(x, y_i)$ for all $(x, y_i) \in X \times Y_i$.

Define $\Gamma_i: X \times Y_i \to 2^{X_i}$ by

$$\Gamma_i(x, y_i) = \begin{cases} s_i(x, y_i), & \forall (x, y_i) \in W_i, \\ X_i, & \forall (x, y_i) \notin W_i. \end{cases}$$

It is easy to show that Γ_i has closed graph in $X \times Y_i \times X_i$, and therefore upper semi-continuous as X_i is compact. Let $\Gamma: X \times Y \to 2^{X \times Y}$ be defined by

$$\Gamma(x, y) := \left(\prod_{i \in I} \Gamma_i(x, y_i), \prod_{i \in I} T_i(x) \right), \quad \forall (x, y) \in X \times Y.$$

Then Γ is upper semi-continuous. Since for each $(x, y) \in X \times Y$, $\Gamma(x, y)$ is nonempty, convex and closed, thus by Kakutani-Fan's fixed point theorem there exists $(\bar{x}, \bar{y}) \in X \times Y$ such that $(\bar{x}, \bar{y}) \in \Gamma(\bar{x}, \bar{y})$, i.e. for each $i \in I$, $\bar{x}_i \in \Gamma_i(\bar{x}, \bar{y}_i)$ and $\bar{y}_i \in T_i(\bar{x})$.

Note that for each $i \in I$, if $(\bar{x}, \bar{y}_i) \in W_i$, then

$$\bar{x}_i = s_i(\bar{x}, \bar{y}_i) \in \varphi_i(\bar{x}, \bar{y}_i) = \mathrm{co}(P_i(\bar{x}, \bar{y}_i)).$$

Thus there exists a finite subset

$$\{x_{i_1}, x_{i_2}, \ldots, x_{i_n}\} \subset P_i(\bar{x}, \bar{y}_i)$$

such that

$$\bar{x}_i \in \mathrm{co}\{x_{i_1}, x_{i_2}, \ldots, x_{i_n}\}$$

and

$$f_i(\bar{x}, x_{i_j}, \bar{y}_i) \in -\mathrm{int}\ C_i(\bar{x}_i), \quad \forall j = 1, 2, \ldots, n.$$

This is a contradiction to (1). Hence $(\bar{x}, \bar{y}_i) \notin W_i$ and so for all $i \in I$, $\varphi_i(\bar{x}, \bar{y}_i) = \emptyset$. Therefore, $\mathrm{co}(P_i(\bar{x}, \bar{y}_i)) = \emptyset$ which implies $P_i(\bar{x}, \bar{y}_i) = \emptyset$. Consequently, we have $\bar{x} \in X$ and $\bar{y}_i \in T_i(\bar{x})$ for each $i \in I$, such that

$$f_i(\bar{x}, z_i, \bar{y}_i) \notin -\mathrm{int}\ C_i(\bar{x}_i), \quad \forall z_i \in X_i.$$

We conclude that SWGVVI has a solution, and this completes the proof. □

Remark 3.1

(1) Contrary to most other papers, it is noted that we do not require the sets $C_i(x_i)$ to be a cone in the present paper.

(2) Note that if for each $i \in I$, the function f_i is continuous and the set-valued map $W_i : X_i \to 2^{E_i}$, defined by $W_i(x_i) := Y_i \backslash \{-\text{int } C_i(x_i)\}$, is upper semi-continuous, then the set $\{(x, y_i) \in X \times Y_i)$: $f_i(x, z_i, y_i) \in -\text{int } C_i(x_i)\}$ is open for all $z_i \in X_i$ (see Zhou and Chen [19]).

Remark 3.2 The following example shows that without the partial diagonally quasi-convexity property of the function $f_i(x, z_i, y_i)$ the corresponding SWGVVI problem may not admit a solution.

Example Let the index set be $I = \{1, 2\}$. For each $i \in I$, let $Y_i = \{y\}$ be a single point in $\mathbb{R}$, and $E_i = X_i = \mathbb{R}$ so that $X = \mathbb{R}^2$. Further, for each $i \in I$ and for all $x_i \in X_i$, let $C_i(x_i) = [0, +\infty)$ and $T_i(x) = \{y\}$ for all $x \in X$. The functions $f_i: X \times X_i \times Y_i \to E_i$, defined by

$$f_i(x, z_i, y) = -\left(\sum_{j=1, j\neq i}^{2} x_j - y\right)^2 + (x_i - z_i) - 1, \qquad i = 1, 2$$

satisfy all the hypothesis of Theorem 3.1 except for the partial diagonally quasi-convexity property. It is not difficult to see that the corresponding SWGVVI problem has no solutions.

Corollary 3.1 *For each $i \in I$, let X_i and Y_i be nonempty compact convex metrizable subsets of two locally convex Hausdorff topological vector spaces, and E_i be a Hausdorff topological vector space. Let $f_i: X \times X_i \times Y_i \to E_i$ be a trifunction and $C_i: X_i \to 2^{E_i}$ be a set-valued map with* int $C_i(x_i) \neq \emptyset$ *for all $x_i \in X_i$, and let $T_i: X \to 2^{Y_i}$ be a upper semi-continuous map with nonempty closed convex values.*

Assume that the following conditions are satisfied.

(1) *For each $i \in I$, for all $x \in X$ and $y_i \in Y_i$, the set $\{z_i \in X_i: f_i(x, z_i, y_i) \in -\text{int } C_i(x_i)\}$ is convex.*

(2) *For each $i \in I$ and for all $z_i \in X_i$, the set $\{(x, y_i) \in X \times Y_i : f_i(x, z_i, y_i) \in -\text{int } C_i(x_i)\}$ is open.*

(3) *For all $x = (x^i, x_i) \in X$ and $y_i \in Y_i$, $f(x, x_i, y_i) \notin -\text{int } C(x_i)$.*

Then, there exists $\bar{x} \in X$ and $\bar{y}_i \in T_i(\bar{x})$ for each $i \in I$, such that

$$f_i(\bar{x}, z_i, \bar{y}_i) \notin -\text{int } C_i(\bar{x}_i), \quad \forall z_i \in X_i.$$

Proof Fix $i \in I$ and $y_i \in Y_i$. We need only to show that $f_i(x, z_i, y_i)$ is weak partial diagonally quasi-convex in z_i with respect to C_i. Suppose not, then there exist $\Lambda = \{x_{i_1}, x_{i_2}, \ldots, x_{i_n}\} \subset X_i$ and $\bar{x} = (\bar{x}^i, \bar{x}_i) \in X$ with $\bar{x}_i \in \text{co } \Lambda$ such that

$$f_i(\bar{x}, x_{i_j}, y_i) \in -\text{int } C_i(\bar{x}_i), \quad \forall j = 1, 2, \ldots, n.$$

Therefore,

$$x_{i_j} \in \{z_i \in X_i: f_i(\bar{x}, z_i, y_i) \in -\text{int } C_i(\bar{x}_i)\} := U_i, \quad \forall j = 1, 2, \ldots, n.$$

By condition (1), the set U_i is convex, and so $\bar{x}_i \in U_i$, i.e.

$$f(\bar{x}, \bar{x}_i, y_i) \in -\text{int } C_i(\bar{x}_i),$$

which is a contradiction to (3). The corollary is proved. □

Remark 3.3 Condition (3) of Corollary 3.1 is satisfied, for example, if for all $x = (x^i, x_i) \in X$ and $y_i \in Y_i, f(x, x_i, y_i) = 0$.

Theorem 3.2 *For each $i \in I$, let X_i and Y_i be nonempty metrizable subsets of two locally convex Hausdorff topological vector spaces, and E_i be a Hausdorff topological space. Let $f_i : X \times X_i \times Y_i \to E_i$ be a trifunction, $C_i : X_i \to 2^{E_i}$ be a set-valued map with* int $C_i(x_i) \neq \emptyset$ *for all $x_i \in X_i$, and T_i: $X \to 2^{Y_i}$ be a upper semi-continuous map with nonempty closed convex values.*

Assume that the following conditions are satisfied.

(1) *For each $i \in I$ and for all $y_i \in Y_i$, $f_i(x, z_i, y_i)$ is weak partial diagonally quasi-convex in z_i with respect to C_i.*
For each $i \in I$ and for all $z_i \in X_i$, the set

$$\{(x, y_i) \in X \times Y_i : f_i(x, z_i, y_i) \in -\text{int } C_i(x_i)\}$$

is open.

(2) *For each $i \in I$, there exist a nonempty compact convex subset K_i of X_i and $\bar{z}_i \in K_i$ such that for each $x \in X \backslash K := X \backslash (\prod_{i \in I} K_i)$,*

$$f_i(x, \bar{z}_i, y_i) \in -\text{int } C_i(x_i), \quad \forall y_i \in T_i(x).$$

Then, there exist $\bar{x} \in K$, and $\bar{y}_i \in T_i(\bar{x})$ for each $i \in I$, such that

$$f_i(\bar{x}, z_i, \bar{y}_i) \notin -\text{int } C_i(\bar{x}_i), \quad \forall z_i \in X_i.$$

Proof For each $i \in I$ and for all $z_i \in X_i$, let

$$G(z_i) := \{x \in K : f_i(x, z_i, y_i) \notin -\text{int } C_i(x_i) \text{ for some } y_i \in T_i(x)\}.$$

We verify first that for each $i \in I$, $G(z_i)$ is closed for all $z_i \in X_i$. Indeed, let $x(\lambda)$ be a net in $G(z_i)$ such that $x(\lambda) \to x^* \in X$, that is, for each $i \in I$, $x(\lambda)_i \to x_i^* \in X_i$. Then there exists $y(\lambda)_i \in T_i(x(\lambda))$ satisfying

$$f_i(x(\lambda), z_i, y_i(\lambda)) \notin -\text{int } C_i(x(\lambda)).$$

Since K is compact and T_i is upper semi-continuous, $T_i(K)$ is therefore compact. Thus we may assume, by passing to a subnet if necessarily, that $y(\lambda)_i$ converges with limit y_i^*. Since T_i is upper semi-continuous and therefore has closed graph, we have $y_i^* \in T_i(x^*)$. By condition (2), we have $f_i(x^*, z_i, y_i^*) \notin -\text{int } C_i(x_i^*)$. Hence $x^* \in G(z_i)$, proving that $G(z_i)$ is closed in X.

For each $i \in I$, let $\{z_{i_1}, \ldots, z_{i_n}\}$ be a finite subset of X_i. Let $A_i = \text{co}(K_i \cup \{z_{i_1}, \ldots, z_{i_n}\})$. Then for each $i \in I$, A_i is a compact convex metrizable subset. By Theorem 3.1, there exist $\bar{x} \in A$ and $\bar{y}_i \in T_i(\bar{x})$ for each $i \in I$ such that

$$f_i(\bar{x}, z_i, \bar{y}_i) \notin -\text{int } C_i(\bar{x}_i), \quad \forall z_i \in A_i.$$

It follows from condition (3) that $\bar{x} \in K$. In particular, we have

$$f_i(\bar{x}, z_{i_k}, \bar{y}_i) \notin -\text{int } C_i(\bar{x}_i), \quad \forall k = 1, 2, \ldots, n$$

proving $\bar{x} \in \cap_{k=1}^n G(z_{i_k})$. Hence, every finite subfamily of $\{G(z_i)\}$ has nonempty intersection. K being compact, so for each $i \in I$ we have $\cap_{z_i \in X_i} G(z_i) \neq \emptyset$. Thus, there exist $\bar{x} \in K$ and $\bar{y}_i \in T_i(\bar{x})$ for each $i \in I$, such that

$$f_i(\bar{x}, z_i, \bar{y}_i) \notin -\text{int } C_i(\bar{x}_i).$$

This completes the proof of the theorem. □

Now we state below existence results of solutions for the system of strong generalized vector variational inequalities.

Theorem 3.3 *For each $i \in I$, let X_i and Y_i be nonempty metrizable subsets of two locally convex Hausdorff topological vector spaces, and E_i be a Hausdorff topological vector space. Let $f_i: X \times X_i \times Y_i \to E_i$ be a trifunction, $C_i : X_i \to 2^{E_i}$ be a set-valued map with $C_i(x_i) \neq \emptyset$ for all $x_i \in X_i$, and $T_i: X \to 2^{Y_i}$ be a upper semi-continuous map with nonempty closed convex values.*

Assume that the following conditions are satisfied.

(1) *For each $i \in I$ and for all $y_i \in Y_i$, $f_i(x, z_i, y_i)$ is strong partial diagonally quasi-convex in z_i with respect to C_i.*
(2) *For each $i \in I$ and for all $z_i \in X_i$, the set*

$$\{(x, y_i) \in X \times Y_i: f_i(x, z_i, y_i) \notin C_i(x_i)\}$$

is open.

Then, there exist $\bar{x} \in X$, and $\bar{y}_i \in T_i(\bar{x})$ for each $i \in I$, such that

$$f_i(\bar{x}, z_i, \bar{y}_i) \in C_i(\bar{x}_i), \quad \forall z_i \in X_i.$$

Theorem 3.4 *For each $i \in I$, let X_i and Y_i be nonempty metrizable subsets of two locally convex Hausdorff topological vector spaces, and E_i be a Hausdorff topological vector space. Let $f_i: X \times X_i \times Y_i \to E_i$ be a trifunction, $C_i : X_i \to 2^{E_i}$ be a set-valued map with $C_i(x_i) \neq \emptyset$ for all $x_i \in X_i$, and $T_i: X \to 2^{Y_i}$ be a upper semi-continuous map with nonempty closed convex values.*

Assume that the following conditions are satisfied.

(1) *For each $i \in I$ and for all $y_i \in Y_i$, $f_i(x, z_i, y_i)$ is strong partial diagonally quasi-convex in z_i with respect to C_i.*
(2) *For each $i \in I$ and for all $z_i \in X_i$, the set*

$$\{(x, y_i) \in X \times Y_i: f_i(x, z_i, y_i) \notin C_i(x_i)\}$$

is open.
(3) *For each $i \in I$, there exist a nonempty compact convex subset K_i of X_i and $\bar{z}_i \in K_i$ such that, for each $x \in X \backslash K := X \backslash (\prod_{i \in I} K_i)$*

$$f_i(x, \bar{z}_i, y_i) \notin C_i(x_i), \quad \forall y_i \in T_i(x).$$

Then, there exist $\bar{x} \in K$ and $\bar{y}_i \in T_i(\bar{x})$ for each $i \in I$, such that

$$f_i(\bar{x}, z_i, \bar{y}_i) \in C_i(\bar{x}_i), \quad \forall z_i \in X_i.$$

The proof of these two theorems, being similar to those of Theorems 3.1 and 3.2, are omitted.

4 Applications

We now derive some existence results for some of the systems of vector variational inequalities that were considered by others.

Case I System of vector equilibrium problems.

The system of vector equilibrium problems, as introduced by Ansari et al. [4], is to find $\bar{x} \in X$ such that for each $i \in I$

$$g_i(\bar{x}, z_i) \notin -\text{int } C, \quad \forall z_i \in X_i,$$

where $g_i: X \times X_i \to E$ is a given bifunction taking values in a Hausdorff topological vector space E, and C is a nonempty pointed closed convex cone in E with $\text{int } C \neq \emptyset$.

Theorem 4.1 *For each $i \in I$, let X_i be a nonempty compact convex metrizable subset of a locally convex Hausdorff topological space, Z be a Hausdorff topological vector space and $C \subset Z$ be a pointed cone with* $\text{int } C \neq \emptyset$. *Let $g_i: X \times X_i \to Z$ be a bifunction.*

Assume that the following conditions are satisfied.

(1) *For each $i \in I$, $g_i(x, z_i)$ is weak partial diagonally quasi-convex in z_i with respect to C.*
(2) *For each $i \in I$ and for all $z_i \in X_i$, the set $\{x \in X : g_i(x, z_i) \in -\text{int } C\}$ is open.*

Then, there exists $\bar{x} \in X$ such that for each $i \in I$,

$$f_i(\bar{x}, z_i) \notin -\text{int } C, \quad \forall z_i \in X_i.$$

Proof This follows directly from Theorem 3.1 by setting for each $i \in I$, $Y_i = \{y\}$ a singleton, $E_i = Z$, $C_i(x_i) = C$ for all $x_i \in X_i$, $T_i(x) = \{y\}$ for all $x \in X$, and $f_i(x, z_i, y_i) = g_i(x, z_i)$ for all $(x, z_i) \in X \times X_i$. □

Remark 4.1 It is easy to see that Theorem 4.1 improves Theorem 2.1 in Ansari et al. [4] under the additional assumptions that each E_i is a locally convex space and X_i is metrizable. Indeed, we note that:

(1) The conditions (see [4]) that C is a pointed closed convex cone with $\text{int } C \neq \emptyset$, for all $x \in X$ the function $z_i \mapsto g_i(x, z_i)$ is C-quasi-convex together with $g(x, x_i) = 0$ for all $x = (x^i, x_i)$, imply that $g_i(x, z_i)$ is weak partial diagonally quasi-convex in z_i with respect to C.
(2) Also when f_i is continuous as assumed in [4], it clearly implies that the set $\{x \in X: g_i(x, z_i) \in -\text{int } C\}$ is open for all $z_i \in X_i$.

Case II System of generalized variational inequalities.

The system of generalized variational inequalities, as studied by Ansari and Yao [3], is to find $\bar{x} \in X$ and $\bar{y}_i \in T_i(\bar{x})$ for each $i \in I$, satisfying

$$\varphi_i(\bar{x}_i, z_i, \bar{y}_i) \geq 0, \quad \forall z_i \in X_i.$$

Here for each $i \in I$, $T_i: X \to 2^{D_i^*}$ is a set-valued map where D_i^* is a nonempty subset of X_i^*, the topological dual of X_i, and $\varphi_i: X_i \times X_i \times D_i^* \to \mathbb{R}$ is a real-valued function.

Theorem 4.2 *For each $i \in I$, let X_i be a nonempty compact convex and metrizable subset of a locally convex Hausdorff topological vector space, D_i^* be a nonempty compact convex and metrizable subset in X_i^*, and $T_i: X \to 2^{D_i^*}$ be a upper semi-continuous set-valued map with nonempty closed convex values. Let $\varphi_i: X_i \times X_i \times D_i^* \to \mathbb{R}$ satisfy the following conditions:*

(1) *For each $i \in I$ and for all $y_i \in D_i^*$, $\varphi_i(x_i, z_i, y_i)$ is 0-diagonally convex in z_i.*
(2) *For each $i \in I$ and for all $z_i \in X_i$, the set*

$$\{(x_i, y_i) \in X_i \times D_i^* : \varphi_i(x_i, z_i, y_i) < 0\}$$

is open.

Then, there exist $\bar{x} \in X$ and $\bar{y}_i \in T_i(\bar{x})$ for each $i \in I$, satisfying

$$\varphi_i(\bar{x}_i, z_i, \bar{y}_i) \geq 0, \quad \forall z_i \in X_i.$$

Proof This follow from Theorem 3.1 by setting for each $i \in I$, $Y_i = D_i^*$, $C_i(x_i) = \mathbb{R}^+$ for all $x_i \in X_i$, and $f_i(x, z_i, y_i) = \varphi_i(x_i, z_i, y_i)$. We need only to note that, if for each $i \in I$ and $y_i \in D_i^*$, $\varphi_i(x_i, z_i, y_i)$ is 0-diagonally convex in z_i, then $f_i(x, z_i, y_i)$ is weak partial diagonally convex in z_i with respect to $\mathbb{R}^+$. □

Remark 4.2 Theorem 4.2 improves Theorem 2.1 of Ansari and Yao [3] under the additional assumptions that for each $i \in I$, X_i is metrizable, D_i^* is compact convex and metrizable, and T_i has nonempty closed convex values.

Case III System of generalized vector variational-like inequalities.

For each $i \in I$, let E_i be a Hausdorff topological vector space, and let X_i and Y_i be nonempty subsets of the locally convex Hausdorff topological vector space S_i and its dual S_i^*, respectively. Let $C_i : X_i \to 2^{E_i}$ be a set-valued map, $L(S^i, E_i)$ be the space of all continuous linear mappings form S_i to E_i, and $< L(S_i, E_i), S_i >$ be a dual system of $L(S_i, E_i)$ and S_i. Given $\theta_i : X \times Y_i \to L(S_i, E_i)$, $\eta_i : X_i \times X_i \to S_i$, and a set-valued map $T_i : X \to 2^{Y_i}$, the system of weak generalized vector variational-like inequalities is to find $\bar{x} \in X$ and $\bar{y}_i \in T_i(\bar{x})$ for each $i \in I$, such that

$$\langle \theta_i(\bar{x}, \bar{y}_i), \eta_i(z_i, \bar{x}_i) \rangle \notin -\text{int}\, C_i(\bar{x}_i), \quad \forall z_i \in X_i.$$

Theorem 4.3 *For each $i \in I$, let E_i be a Hausdorff topological vector space, and let X_i, Y_i be nonempty metrizable subsets of the locally convex Hausdorff topological vector space S_i and its dual S_i^*, respectively. Let $C_i : X_i \to 2^{E_i}$ be a set-valued map such that for all $x_i \in X_i$, $C_i(x_i)$ is a proper closed convex cone in E_i with* int $C_i(x_i) \neq \emptyset$. *Let the set-valued map $T_i : X \to 2^{Y_i}$ be upper semi-continuous with nonempty closed convex values.*

Assume that the functions $\theta_i : X \times Y_i \to L(S_i, E_i)$ and $\eta_i : X_i \times X_i \to S_i$ satisfy the following conditions:

(1) *For each $i \in I$, $\eta_i : X_i \times X_i \to S_i$ is affine in the first argument such that $\eta_i(x_i, x_i) = 0$, $\forall x_i \in X_i$.*
(2) *For each $i \in I$ and for all $z_i \in X_i$, the set*

$$\{(x, y_i) \in X \times Y_i : \langle \theta_i(x, y_i), \eta_i(z_i, x_i) \rangle \in \text{int}\, C_i(x_i)\}$$

is open.
(3) *For each $i \in I$, there exists a nonempty compact convex subset K_i of X_i and $\bar{z}_i \in K_i$ such that, for each $x \in X \backslash K := X \backslash (\prod_{i \in I} K_i)$,*

$$\langle \theta_i(x, y_i), \eta_i(\bar{z}_i, x_i) \rangle \in -\text{int}\, C_i(x_i), \quad \forall y_i \in T_i(x)$$

Then, there exist $\bar{x} \in X$ and $\bar{y}_i \in T_i(\bar{x})$ for each $i \in I$, such that

$$\langle \theta_i(\bar{x}, \bar{y}_i), \eta_i(z_i, \bar{x}_i) \rangle \notin -\text{int}\, C_i(\bar{x}_i), \quad \forall z_i \in X_i.$$

Proof This result follows from Theorem 3.2 by setting for each $i \in I$,

$$f_i(x, z_i, y_i) = \langle \theta_i(x, y_i), \eta_i(z_i, x_i) \rangle.$$

Letting $I = \{1\}$ in Theorem 4.3 leads to the existence result of solutions for this generalized vector variational-like inequalities as studied in [5,10]. □

Corollary 4.1 *Let X and Y be nonempty metrizable subsets of the locally convex Hausdorff topological vector space S and its dual S^*, respectively. Let E be a Hausdorff topological vector space, and $C: X \to 2^E$ be a set-valued map such that for all $x \in X$, $C(x)$ is a proper closed convex cone in E with* $\operatorname{int} C(x) \neq \emptyset$. *Let the set-valued map $T: X \to 2^Y$ be upper semi-continuous with nonempty closed convex values. Assume the functions $\theta: X \times Y_i \to L(S, E)$ and $\eta: X \times X \to S$ satisfy the following conditions.*

(1) *Let $\eta_i: X_i \times X_i \to S_i$ be affine in the first argument such that $\eta_i(x_i, x_i) = 0$, $\forall x_i \in X_i$.*
(2) *For $z \in X$ the set*

$$\{(x, y) \in X \times Y_i : \langle \theta_i(x, y), \eta_i(z, x) \rangle \in \operatorname{int} C_i(x_i)\}$$

is open.
(3) *There exist a nonempty compact convex subset K of X and $\bar{z} \in K$ such that, for each $x \in X \backslash K$,*

$$\langle \theta_i(x, y), \eta_i(\bar{z}, x) \rangle \in -\operatorname{int} C(x), \quad \forall y \in T(x).$$

Then, there exist $\bar{x} \in X$ and $\bar{y} \in T(\bar{x})$ such that

$$\langle \theta(\bar{x}, \bar{y}), \eta(z, \bar{x}) \rangle \notin -\operatorname{int} C(\bar{x}), \quad \forall z \in X.$$

Remark 4.3 Corollary 4.1 improves Theorem 3 of Ansari et al. [5] under the additional assumptions that S is locally convex and T is upper semi-continuous with nonempty closed convex values.

We also have the following result directly.

For each $i \in I$, let $A_i: X \to L(X_i, Y_i)$ be a given mapping, where $L(X_i, Y_i)$ denotes the space of all continuous linear operators from X_i into Y_i. The system of vector variational inequalities (SVVI) with variable preference is to find $\bar{x} \in X$ such that for each $i \in I$,

$$\langle A_i(\bar{x}), y_i - \bar{x}_i \rangle \notin -\operatorname{int} C_i(\bar{x}_i), \quad \forall y_i \in X_i.$$

Corollary 4.2 *For each $i \in I$, let X_i be a nonempty convex metrizable subset of a locally convex Hausdorff topological vector space E_i, let $A_i : X \to L(X_i, Y_i)$ be continuous on X, and let $C_i: X_i \to 2^{Y_i}$ be a set-valued map such that $C_i(x_i)$ is a convex cone with* $\operatorname{int} C_i(x_i) \neq \emptyset$ *for all $x_i \in X_i$. Suppose further that the set-valued map $W_i: X_i \to 2^{Y_i}$, defined by $W_i(x_i) := Y_i \backslash -\operatorname{int} C_i(x_i)$ for all $x_i \in X_i$, is upper semi-continuous, and for each $i \in I$ there exists a nonempty compact convex subset K_i of X_i and $\bar{y}_i \in K_i$ such that for all $x = (x^i, x_i) \in X \backslash K =: X \backslash \prod_{i \in I} K_i$,*

$$\langle A_i(x), \bar{y}_i - x_i \rangle \in -\operatorname{int} C_i(x_i).$$

Then, there exists a solution $\bar{x} \in X$ for SVVI.

Acknowledgments This research was supported by the Research Committee of The Hong Kong Polytechnic University and by National Natural Science Foundation of China(No.70501015).

References

1. Ansari, Q.H., Yao, J.C.: A fixed-point theorem and its application to the system of variational inequalities. Bull. Aust. Math. Soc. **59**, 433–442 (1999)
2. Ansari, A.H.: Vector equilibrium problems and vector variational inequalities. In: Giannessi, G. (ed.) Vector Variational Inequalities and Vector Equilibria., pp. 1–15. Kluwer Academic Publishers, Dordrecht (2000)
3. Ansari, Q.H., Yao, J.C.: Systems of generalized inequalities and their applications. Appl. Anal. **76**, 203–217 (2000)
4. Ansari, Q.H., Schaible, S., Yao, J.C.: System of vector equilibrium problems and its applications. J. Optim. Theory Appl. **107**, 547–557 (2000)
5. Ansari, Q.H., Siddiqi, A.H., Yao, J.C.: Generalized vector variational-like inequalities and their scalarization. In: Giannessi, G. (ed.) Vector Variational Inequalities and Vector Equilibria, pp. 17–37. Kluwer Academic Publishers, Dordrecht (2000)
6. Bianchi M.: Pseudo P-monotone operators and variational inequalities, Report 6. Istituto di Econometria e Matematica per le Decisioni Economiche Universita Cattolica del Sacro Cuore, Milan, Italy (1993)
7. Chadli, O., Riahi, H.: On generalized vector equilibrium problems. J. Glob. Optim. **18**, 33–41 (2000)
8. Chen, G.Y. , Hou, S.H.: Existence of solutions for vector variational inequalities. In: Giannessi, G. (ed.) Vector Variational Inequalities and Vector Equilibria., pp. 73–86. Kluwer Academic Publishers, Dordrecht (2000)
9. Cohen, G., Chaplais, F.: Nested monotony for variational inequalities over a product of spaces and convergence of iterative algorithms. J. Optim. Theory Appl. **59**, 369–390 (1988)
10. Ding, X.P., Tarafdar, E.: Generalized vector variational-like inequalities. In: Giannessi, G. (ed.) Vector Variational Inequalities and Vector Equilibria., pp. 113–124. Kluwer Academic Publishers, Dordrecht (2000a)
11. Ding, X.P., Tarafdar, E.: Generalized vector variational-like inequalities with $C_x - \eta -$ pseudomonotone set-valued mappings. In: Giannessi, G. (ed.) Vector Variational Inequalities and Vector Equilibria., pp. 125–140. Kluwer Academic Publishers, Dordrecht (2000b)
12. Klein, E., Thompson, A.C.: Theory of Correspondences. Wiley, New York (1984)
13. Konnov, I.V.: On system of variational inequalities. Russ. Math. (Iz. VUZ). **41**, 79–88 (1997)
14. Lai, T.C., Yao, J.C.: Existence results for VVIP. Appl. Math. Lett. **9**(3), 17–19 (1996)
15. Lee, G.M., Kim, D.S., Lee, B.S.: Generalized vector variational inequality. Appl. Math. Lett. **9**(1), 39–42 (1996)
16. Pang, J.S.: Asymmetric variational inequality problems over product sets: applications and iterative methods. Math. Program. **31**, 206–219 (1985)
17. Qun, L.: Generalized vector variational-like inequalities. In: Giannessi, G. (ed.) Vector Variational Inequalities and Vector Equilibria., pp. 363–369. Kluwer Academic Publishersm, Dordrecht (2000)
18. Siddiqi, A.H., Ansari, Q.H., Khaliq, A.: On vector variational inequalities. J. Optim. Theory Appl. **84**(1), 171–180 (1995)
19. Zhou, J.X., Chen, G.: Diagonal convexity conditions for problems in convex analysis and quasi-variational inequalities. J. Math. Anal. Appl. **132**, 213–225 (1988)